

# KP Hierarchies, Polynomial and Rational $\mathcal{W}$ Algebras On Riemann Surfaces: A Global Approach.

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## Abstract

A covariant pseudodifferential calculus on Riemann surfaces, based on the Krichever-Novikov global picture, is presented. It allows defining scalar and matrix KP operators, together with their reductions, in higher genus. Globally defined Miura maps are considered and the arising of polynomial or rational  $\mathcal{W}$  algebras on R.S. associated to each reduction are pointed out. The higher genus NLS hierarchy is analyzed in detail.

# 1 Introduction.

It is very well known that a deep connection exists between the algebraic geometry and the integrable partial differential equations of the soliton theory. Such connection has been developed since the seventies, from the work of, among others, Novikov, Dubrovin, Matveev, Krichever, [1]; for a more complete historical account see [2] and references therein.

In such a framework, periodic or quasi-periodic solutions are constructed with the help of the Baker-Akhiezer functions defined on auxiliary Riemann surfaces. Roughly speaking, the solutions are obtained from “spectral” algebraic geometrical data which, in a generic case, correspond to a bundle over the moduli space  $M_{g,N}$  of smooth algebraic curves of genus  $g$  with  $N$  punctures. So for instance, in the case of the KP hierarchy, the solutions are obtained for the bundles over the moduli space with 1 puncture ( $N = 1$ ).

Another possibility of connecting Riemann surfaces with integrable equations of non-linear type has been investigated in [3]. In this work the KdV equation was covariantized in order to describe a (euclidean) dynamics over a specified Riemann surface  $\Sigma$ , providing in such a way the generalization of the classical KdV equation on a cylinder (which can be conformally mapped to describe a dynamics over the punctured complex plane  $C^*$ ). This work made use of the global description of a Riemann surface obtained by replacing the Fourier-Laurent bases with the Krichever-Novikov ones [4] relative to the given Riemann surface.

In this paper I will reconsider the scheme pioneered in [3] in order to formulate in the global Krichever-Novikov framework not only the KdV or standard Drinfeld-Sokolov type of hierarchies[5], but the whole KP hierarchy, together with its non-standard (i.e. non-principal in the  $gl(n)$ -matricial language,[6]) reductions (see also [7]). Such reductions lead to the coset hierarchies like the Non-Linear Schrödinger equation, and are associated to  $\mathcal{W}$  algebras of rational type [8, 9]. To reach this goal one needs to refine the set of “covariantization rules” proposed in [3]: basically one needs to covariantize the whole pseudodifferential operator calculus.

A remark is in order: in this framework one has the freedom to choose among different dynamics: if we do not force the time variable being identified with the euclidean time  $\tau$  on a Riemann surface, then the evolution equation corresponds to the real time evolution of a field defined over a punctured Riemann surface  $\Sigma^*$  (here  $\Sigma^* = \Sigma \setminus P_{\pm}$ , with  $P_{\pm}$  corresponding to the North and South Pole of the Riemann surface). Such dynamics is not uniquely defined: every holomorphic one-form on  $\Sigma^*$  defines its corresponding dynamics. This degeneracy can be removed with a canonical choice however if the evolution flow corresponds to the flow along the euclidean time, leading to a nice geometrical interpretation. It should be pointed out however that the identification of the evolution parameter with the euclidean time is mandatory for relativistic theories like the WZNW models on Riemann surfaces; on the contrary it is not so essential for non-relativistic systems like those described by the KP hierarchy and its reductions.

Some useful results concerning the covariantization of conformal operators (Bol operators and so on) can be found in [10]. Recently in a couple of papers [11] the formulation of Drinfeld-Sokolov hierarchies on Riemann surfaces and the appearance of  $\mathcal{W}$  algebras in the light of the Krichever-Novikov framework have been discussed. In these papers only

the principal reductions have been considered.

The generalization of the construction here presented to the case of Riemann surfaces with more punctures can be straightforwardly performed as well by using the results obtained in a series of papers by Schlichenmaier (see e.g; [12]).

The scheme of the paper is the following: In the next section the needed mathematical framework will be introduced. I will follow the approach and results of [13], where in particular some useful results concerning the Heisenberg representation of the KN algebra were first obtained. Next, the pseudodifferential calculus will be introduced for Riemann surfaces. KP hierarchies and their reductions will be conveniently formulated. The Poisson brackets structure (leading to  $\mathcal{W}$  algebras) will be provided through (higher genus) Miura maps.

In the fourth section the case of non-principal reductions of the matrix KP hierarchy on Riemann surfaces will be considered by analyzing in detail as an example the Non-Linear-Schrödinger equation. The free-field Wakimoto representation will be realized in the higher genus case and the appearance of a rational  $\mathcal{W}$  algebra (see [8]) will be discussed.

## 2 Notations and conventions.

On a Riemann surface  $\Sigma$  of genus  $g$  let us consider two distinguished points  $P_+$  and  $P_-$  and local coordinates  $z_+$  and  $z_-$  around them, such that  $z_{\pm}(P_{\pm}) = 0$ . On  $\Sigma$  I will consider bases of meromorphic tensors which are holomorphic except at  $P_{\pm}$ , in particular of meromorphic vector fields  $e_I$ , functions  $A_I$ , 1-differentials  $\omega^I$  and quadratic differentials  $\Omega^I$ . Here  $I$  is integer or half-integer according to whether  $g$  is even or odd. The behaviour near  $P_{\pm}$  is given by

$$A_I(z_{\pm}) = a_I^{\pm} z_{\pm}^{\pm I - \frac{g}{2}} (1 + \mathcal{O}(z_{\pm})) \quad (1)$$

$$\omega^I(z_{\pm}) = b_I^{\pm} z_{\pm}^{\mp I + \frac{g}{2} - 1} (1 + \mathcal{O}(z_{\pm})) (dz_{\pm}) \quad (2)$$

$$e_I(z_{\pm}) = c_I^{\pm} z_{\pm}^{\pm I - g_0 + 1} (1 + \mathcal{O}(z_{\pm})) \frac{\partial}{\partial z_{\pm}}, \quad g_0 = \frac{3}{2}g \quad (3)$$

$$\Omega^I(z_{\pm}) = d_I^{\pm} z_{\pm}^{\mp I + g_0 - 2} (1 + \mathcal{O}(z_{\pm})) (dz_{\pm})^2 \quad (4)$$

For  $|I| \leq \frac{g}{2}$  the definitions (1,2) must be modified, because of the Weierstrass theorem. Let us set  $A_{\frac{g}{2}} = 1$ , while for  $I = \frac{g}{2} - 1, \dots, -\frac{g}{2}$  the power of  $z_-$  is lowered by 1 in (1). As for  $\omega^I$  and  $I = \frac{g}{2} - 1, \dots, -\frac{g}{2}$  the power of  $z_-$  must be raised by 1 in (2) and  $\omega^{\frac{g}{2}}$  is set equal to the third kind differential

$$\omega^{\frac{g}{2}}(z_{\pm}) = \pm \frac{1}{z_{\pm}} [1 + \mathcal{O}(z_{\pm})] (dz_{\pm}) \quad (5)$$

normalized in such a way that the periods around any cycle are purely imaginary.

The above bases elements are determined up to numerical constants due to the Riemann-Roch theorem. So let us set e.g.  $a_I^+ = 1$ , then the  $a_I^-$ 's are completely determined. One

can do the same for the  $c_I^\pm$ 's. As for the other constants they are fixed by the duality relations

$$\begin{aligned}\frac{1}{2\pi i} \oint_{C_\tau} dQ A_I(Q) \omega^J(Q) &= \delta_I^J \\ \frac{1}{2\pi i} \oint_{C_\tau} dQ e_I(Q) \Omega^J(Q) &= \delta_I^J\end{aligned}\tag{6}$$

Here  $C_\tau$  denotes a level curve of the univalent function

$$\tau(Q) = \operatorname{Re} \int_{Q_0}^Q \omega^{\frac{g}{2}}\tag{7}$$

for a fixed  $Q_0 \in \Sigma$ . Of course the integrals of eq.(6) do not change for a contour which can be continuously deformed to a  $C_\tau$ . Henceforth the symbol  $\oint$  without any specification will denote integration around  $C_\tau$ .

The Lie brackets of the bases element  $e_I$  are

$$[e_I, e_J] = C_{IJ}^K e_K,\tag{8}$$

Here and throughout the paper summation over repeated indices is understood, unless otherwise stated. The structure constants  $C_{IJ}^K$  can be calculated from the constants appearing in the expansion of  $e_I$

$$C_{IJ}^K = \frac{1}{2\pi i} \oint [e_I, e_J] \Omega^K$$

Eq. (8) defines what we call the KN algebra over  $\Sigma$ . The central extension of this algebra is defined by means of the cocycle

$$\chi(e_I, e_J) = \frac{1}{24\pi i} \oint \tilde{\chi}(e_I, e_J)$$

the integral is over any contour surrounding  $P_+$ , and  $\tilde{\chi}(f, g)$  for any two meromorphic vector fields  $f = f(z) \frac{\partial}{\partial z}$  and  $g = g(z) \frac{\partial}{\partial z}$ , is given by

$$\tilde{\chi}(f, g) = \left( \frac{1}{2} (f'''g - g'''f) - R(f'g - fg') \right) dz_+$$

$R$  is a Schwarzian connection.

The relation

$$[e_I, e_J] = C_{IJ}^K e_K + t \chi(e_I, e_J), \quad [e_I, t] = 0\tag{9}$$

defines the extended KN algebra.

In the following the extension  $\hat{\mathcal{A}}^\Sigma$  of the commutative algebra  $\mathcal{A}^\Sigma$  of the  $A_I$ 's (generalized Heisenberg algebra), will be needed; it is defined by

$$[A_I, A_J] = \hat{k} \gamma_{IJ}, \quad [A_I, \hat{k}] = 0\tag{10}$$

where

$$\gamma_{IJ} = \frac{1}{2\pi i} \oint A_I dA_J \quad (11)$$

The cocycle  $\gamma_{IJ}$  vanishes for  $|I + J| > g + 1$ , as it is easy to verify from eq.(11).

In the following one also needs the notations

$$N_i^J = \frac{1}{2\pi i} \oint_{a_i} \omega^J \quad (12)$$

$$M_i^J = \frac{1}{2\pi i} \oint_{b_i} \omega^J \quad (13)$$

where  $\{a_i, b_i\}, i = 1, \dots, g$  is a basis of homology cycles. From eq.(11) one gets

$$dA_I = -\gamma_{IJ} \omega^J$$

Integrating this equation along the homology cycles one obtains

$$0 = \frac{1}{2\pi i} \oint_{a_i} dA_J = N_i^L \gamma_{LJ} \quad (14)$$

$$0 = \frac{1}{2\pi i} \oint_{b_i} dA_J = M_i^L \gamma_{LJ} \quad (15)$$

Thus there are  $2g + 1$  eigenvectors of the matrix  $\gamma$  with 0 eigenvalue, taking into account that  $\gamma_{J\frac{g}{2}} = 0$  as a consequence of  $A_{\frac{g}{2}} \equiv 1$ .

From (14) and (15) one has

$$\begin{aligned} d(N_i^L A_L) &= N_i^L \gamma_{KL} \omega^K = 0 \\ d(M_i^L A_L) &= M_i^L \gamma_{KL} \omega^K = 0 \end{aligned}$$

Therefore  $N_i^L A_L$  and  $M_i^L A_L$  are constant (outside  $P_{\pm}$ ).

The delta function property is introduced through

$$\frac{1}{2\pi i} \oint_{C_{\tau}} \Delta(Q, Q') \psi(Q) = \psi(Q')$$

for any one-form  $\psi$ , where

$$\Delta(Q, Q') = A_J(Q) \omega^J(Q')$$

Finally let me introduce for later convenience the constants  $\alpha^L_{JK}$ , defined through

$$\alpha^L_{JK} = \frac{1}{2\pi i} \oint A_J A_K \omega^L \quad (16)$$

In genus 0 we have

$$\alpha^L_{JK} = \delta_{L, J+K}, \quad \gamma_{JK} = K \delta_{J+k, 0}$$

### 3 The KP hierarchy.

Let us recall the basic features of the KP hierarchy. A convenient way of introducing it makes use of the pseudodifferential operator (PDO) formalism (see [14]): without even referring to any hamiltonian structure, the KP flows can be defined via the pseudodifferential Lax operator  $L$ :

$$L = \partial + \sum_{n=0}^{\infty} u_n \partial^{-n-1} \quad (17)$$

where the  $u_i$  are an infinite set of fields depending on the spatial coordinate  $x$  and the time parameters  $t_k$ . The different flows are defined through the position

$$\frac{\partial L}{\partial t_k} = [L, L^k_+] \quad (18)$$

where  $k$  is a positive integer and  $L^k_+$  denotes the purely differential part of the operator. The quantities

$$F_k = \langle L^k \rangle \quad (19)$$

are first integral of motions for the flows (18) (the symbol  $\langle A \rangle$  denotes the integral of the residue,  $\langle A \rangle = \int dw a_{-1}(w)$  for the generic pseudodifferential operator

$$A = \dots + a_{-1} \partial^{-1} + \dots$$

The basic requirement of the pseudodifferential calculus lays on the commutation rule

$$\partial^{-1} f = f \partial^{-1} + \sum_{r=1}^{\infty} (-1)^r f^{(r)} \partial^{-r-1} \quad (20)$$

(where  $f^{(r)} \equiv \partial^r f$ ), together with the standard properties of the derivative (Leibniz rule, etc.). In order to formulate the KP hierarchy in higher genus,  $L$  should be regarded as a covariant operator, mapping tensors into tensors, and it must be checked that the pseudodifferential calculus with covariant derivatives works as in the standard case.

Let us denote as  $f_\lambda = \overline{f}_\lambda(z)(dz)^\lambda$  a  $\lambda$ -weight tensor (from now on, overlined quantities will always denote the expression in local coordinates). The covariant derivative, mapping tensors of weight  $\lambda$  into tensors of weight  $\lambda + 1$  is introduced through the position

$$\mathcal{D} = dz(\partial_z - \lambda \overline{\Gamma}(z)) \quad (21)$$

where  $\Gamma$  is a connection, transforming under reparametrizations as the logarithmic derivative of a 1-form.

The space of connections is an affine space, which means that every connection can be expressed as the sum of a given logarithmic derivative of reference plus a holomorphic (outside  $P_\pm$ ) 1-form. Even if any connection can be used to form a covariant derivative, nevertheless, the PDO formalism seems inconsistent unless  $\Gamma$  can be expressed as a logarithmic derivative of a 1-form, no matter which<sup>1</sup>. Therefore I will consider the PDO formalism only for the restricted class of connections which can be represented as

$$\overline{\Gamma} = \frac{\partial \overline{\omega}}{\overline{\omega}} \quad (22)$$

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<sup>1</sup>This has to do with the fact that “inverting” a generic covariant derivative leads to non-single valued integrals.

$\omega$  can be chosen arbitrarily; different choices will provide different dynamics. There is however a uniquely defined canonical choice for  $\omega$ : in order to allow the identification of the dynamics with the euclidean evolution on  $\Sigma$  only the choice  $\omega \equiv \omega^{\frac{g}{2}}$  (the abelian differential of the third kind normalized in such a way that its integrals along the homology cycles are purely imaginary) is allowed. This point has been discussed in [3], so that I will not come back any longer on it.

Notice that  $\omega$  with respect to the connection defined out of it is a covariantly constant 1-form, namely it satisfies

$$\mathcal{D}_\Gamma \omega = 0 \quad (23)$$

Therefore  $\omega$  can be used to construct covariantly constant tensors at any order, turning any constant into a covariant constant having the wished tensorial properties.

At this point we have to introduce the “inverse” operator with respect to the covariant derivative. To do that let us introduce the covariant integral  $\int_\Gamma$  associated to the connection  $\Gamma$ ; the covariant integral maps tensors of weight  $\lambda$  into tensors of weight  $\lambda - 1$ . The right choice for  $\int_\Gamma$  is the following:

$$\int_\Gamma = \omega^\lambda(Q) \cdot \int_{Q_0}^Q \omega^{\lambda-1}(\overline{Q}). \quad (24)$$

Here  $Q_0$  is a given reference point, the only requirement being  $Q_0 \neq P_\pm$ . It is a straightforward check to prove that  $\int_\Gamma$  has the right properties: one gets indeed

$$\mathcal{D}_\Gamma \cdot \int_\Gamma = 1 \quad (25)$$

and

$$\int_\Gamma \mathcal{D}_\Gamma f^\lambda(Q) = f^\lambda(Q) - \omega^\lambda(Q) \frac{f^\lambda(Q_0)}{\omega^\lambda(Q_0)} \quad (26)$$

where the last quantity on the r.h.s. is a covariantly constant  $\lambda$ -tensor. With the above definitions covariant tensors and covariant integrals can be used as ordinary derivatives and integrals: in particular it is possible to integrate by parts; moreover the covariant integral of a totally covariant derivative along any closed contour  $\mathcal{C}$  is always vanishing:

$$\oint_{\mathcal{C}, \Gamma} \mathcal{D}_\Gamma f = 0 \quad (27)$$

Therefore it makes sense to introduce the symbol  $\mathcal{D}^{-1}$  and the standard rules of the PDO calculus can be applied as before. Notice that with the above definitions one can take contour integrals for tensors of any order.

From now on, to simplify notation, covariant integrals will be denoted with the same symbol as ordinary integrals: no confusion will arise, the context will say which is which.

The covariant KP hierarchy is defined by replacing the ordinary derivative with the covariant one in (17). Taking into account that  $L^k$  is a  $k$ -th differential, the covariant flows of the KP hierarchy can be introduced through

$$\omega^k \frac{\partial L}{\partial t_k} = [L, L^k_+] \quad (28)$$

(no summation in the l.h.s. of course).

The first integrals of motions are obtained from (19) by replacing the integrals with covariant ones and taking  $C_\tau$  as contour integration. Due to the presence of the poles in  $\frac{1}{\omega}$ , different choices of  $\tau$  lead to different values for the first integrals. Once fixed a particular  $\bar{\tau}$ , the curve  $C_{\bar{\tau}}$  can be regarded on the same foot as the line integrals in the flat case.

Up to now no Poisson brackets structure has been introduced. However once the “covariantization rules” are at disposal, it is quite immediate to introduce the set of Poisson brackets by repeating the steps of [14]. In order to leave this paper at a reasonable letter-size, all that will not be discussed explicitly, rather in the next section the Poisson brackets dynamics will be introduced through the perhaps more interesting framework of matricial KP, and the connection between matricial and scalar KP will be pointed out.

Here let us simply recall that the standard Drinfeld-Sokolov reductions of the KP hierarchy can be obtained by imposing the constraint, consistent with the flows (18)

$$L^n = L^n_+ \equiv L_n \quad (29)$$

(with  $n$  positive integer  $= 2, 3, \dots$ ), which tells that the  $n$ -th power of  $L$  is a purely differential operator. For  $n = 2$  one gets the higher genus KdV equation ( $L_2 = \mathcal{D}^2 + T$ ); for  $n = 3$  the higher genus Boussinesq equation ( $L_3 = \mathcal{D}^3 + U\mathcal{D} + V$ ).

The Poisson brackets structure can be introduced also through free-fields Miura maps, namely by representing the  $n$ -th differential operator through the position

$$L_n = (\mathcal{D} - \Phi_1)(\mathcal{D} - \Phi_2) \dots (\mathcal{D} - \Phi_n) \quad (30)$$

supplemented by the condition  $\Phi_1 + \Phi_2 + \dots + \Phi_n = 0$ . The  $n - 1$  independent fields  $\Phi_i$ ’s are 1-forms satisfying the free-fields algebra on Riemann surfaces, i.e. the higher genus Heisenberg algebra (10):

$$\begin{aligned} \Phi_i(Q) &= \alpha_{J,i} \omega^J(Q) \\ \{\Phi_i(Q), \Phi_j(Q')\} &= -\delta_{ij} \gamma_{IJ} \omega^I(Q) \omega^J(Q') \end{aligned} \quad (31)$$

for  $i, j = 1, \dots, n - 1$ .

The KdV field  $T(Q)$  in  $L_2$  can be represented as

$$T(Q) = \mathcal{D}\Phi(Q) - \Phi^2(Q) \quad (32)$$

It is a straightforward check to show that when  $\Phi(Q)$  is assumed to satisfy the higher genus Heisenberg algebra (10), then the Poisson brackets algebra of the higher genus momenta

$$L_I = \frac{1}{2\pi i} \oint e_I T \quad (33)$$

corresponds to the Krichever-Novikov algebra with central charge  $c = 1$ . The cocycle will be fixed by the choice of the connection in  $\mathcal{D}$ , however different connections will give rise to different cocycles belonging to the same cohomology class. The value of the



central charge is uneffective at the classical level since it can always be rescaled to any non-vanishing value via field redefinitions and Poisson brackets rescalings.

Similarly the two fields  $\Phi_{1,2}$  in  $(\mathcal{D}-\Phi_1)(\mathcal{D}-\Phi_2)(\mathcal{D}+\Phi_1+\Phi_2)$  will give rise a higher genus  $W_3$  algebra structure for the Boussinesq fields  $U$  and  $V$ . The momenta for the 3-tensor  $V$  should be taken with respect to the basis  $f^{-2}_I$  of weight  $-2$  tensors:  $V_I = \frac{1}{2\pi i} \oint f^{-2}_I V$ . When  $T$  (and respectively  $U, V$ ) satisfy the KdV (Boussinesq) equation on Riemann surface, the fields  $\Phi$  ( $\Phi_{1,2}$ ) satisfy the corresponding higher genus mKdV (modified Boussinesq). The extension of this structure to generic values of  $n$  is immediate.

## 4 The NLS Hierarchy from Matrix KP.

In this section I will show how to put on Riemann surfaces the AKS approach to the hierarchies based on matrix-type Lax operators [6] and to connect it with the scalar KP; the scheme here adopted has been presented in [15] and is part of a more developed forthcoming paper on this topic.

One starts with the matrix Lax operator

$$\mathcal{L} = \partial_x + J(x) + \lambda K \quad (34)$$

where  $J(x)$  are currents valued in some finite Lie algebra  $\mathcal{G}$ ,  $\lambda$  is a spectral parameter and  $K$  is a constant element in  $\mathcal{G}$ . The Kac-Moody current algebra  $\hat{\mathcal{G}}$  is one of the Poisson brackets structure for the above system (the one we are interested in). The generalized Drinfeld-Sokolov hierarchies are obtained (see[6]) by assuming  $K$  being a regular element for the loop algebra  $\tilde{\mathcal{G}}$ , with  $\lambda$  as loop parameter. The regularity condition means that

$$\tilde{\mathcal{G}} = Ker K \oplus Im K \quad (35)$$

where the action is the adjoint one.

Under the above assumptions it is possible to introduce a grading  $deg$  on the elements of the loop algebra, defined in such a way that  $deg(\lambda K) = 1$ . Different gradings induce different hierarchies: the principal grading, which associates the grade one to the simple positive roots of the algebra, provides the standard DS hierarchies. Another interesting kind of gradings is furnished by the homogeneous one ( $deg \equiv \frac{d}{d\lambda}$ ). The NLS hierarchy is obtained by taking the homogeneous grading w.r.t. the  $\mathcal{G} \equiv sl(2)$  algebra having  $H$  as Cartan generator and  $E_{\pm}$  as roots. In this case  $K = H$  is a regular element.

The main property of the above operator can be stated as follows: there exists an adjoint transformation  $\mathcal{L} \mapsto \mathcal{L}_{\alpha} = adj(\alpha)\mathcal{L}$ ,

$$\mathcal{L}_{\alpha} = \mathcal{L} + [\alpha, \mathcal{L}] + \frac{1}{2}[\alpha, [\alpha, \mathcal{L}]] + \dots \quad (36)$$

which preserves both the Poisson brackets and the monodromy invariants. Under the condition of regularity for  $K$  it is possible to uniquely determine, with an iterative procedure, the local fields  $\alpha(x) \in Im K$  ( $\alpha(x)$  expanded in the components with negative gradings only), such that the transformed operator  $\mathcal{L}_{\alpha}$  is diagonal:

$$\mathcal{L}_{\alpha} = \partial_x + R(x) + \lambda K \quad (37)$$

where  $R(x) \in \text{Ker} K$  is expanded over the non-positive grading components and can be iteratively computed. The diagonal character of the transformed operator makes possible to compute the monodromy invariants. The different components of  $R(x)$  provide the tower of hamiltonian densities.

To be explicit, in the NLS example we have

$$\alpha(x) = \lambda^{-1}\alpha_1 + \lambda^{-2}\alpha_2 + \dots \quad (38)$$

with

$$\alpha_j = \alpha_{j,+}E_+ + \alpha_{j,-}E_- \quad (39)$$

and

$$R(x) = R_0H + \lambda^{-1}R_1H + \lambda^{-2}R_2H + \dots \quad (40)$$

At the lowest order, with easy computations, we get

$$\begin{aligned} R_0 &= J_0 \\ R_1 &= 2J_+J_- \\ R_2 &= J_+\Delta J_- - J_-\Delta J_+ \end{aligned} \quad (41)$$

(where  $\Delta$  is the covariant derivative w.r.t. the Kac-Moody  $\hat{U}(1)$  subalgebra:  $\Delta J_{\pm} = (\partial \pm 2J_0)J_{\pm}$ , see [8]).  $R_2$  provides the hamiltonian density for the Non-Linear-Schrödinger equation with respect to the Kac-Moody Poisson brackets. The hamiltonians of the infinite tower are all in involution with respect to the Poisson brackets structure. Moreover it is easily shown by induction that every hamiltonian density  $R_i$  with  $i > 0$  has vanishing Poisson brackets with the current  $J_0$ , i.e. it belongs to the  $\hat{U}(1)$  coset.

Consistent reductions of the previously considered Lax operator  $L$  of the scalar KP hierarchy can be recovered from the matrix KP  $\mathcal{L}$  operator with the following procedure: let us take for simplicity the  $sl(2)$  case in the fundamental representation, then if we solve the matrix equation

$$\left( \partial \mathbf{1} + \begin{pmatrix} J_0 & J_+ \\ J_- & -J_0 \end{pmatrix} \right) \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = 0 \quad (42)$$

for, let's say, the  $\Psi_-$  component and allow inverting the derivative operator, we can plug the result into the equation for the  $\Psi_+$  component, obtaining

$$(\Delta + J_-\Delta^{-1}J_+)\Psi_+ = 0 \quad (43)$$

The operator  $L = \Delta + J_-\Delta^{-1}J_+$  provides a consistent reduction of KP (the one associated with the NLS equation, see also [9]). A Poisson brackets structure for  $L$  is induced and coincides with the Kac-Moody Poisson brackets for the matrix KP operator. Analogous steps can be performed in the general case as well (choice of different affine Lie algebras and different representations).

In order to define the matrix KP hierarchy on Riemann surfaces, we have to promote the Lax operator  $\mathcal{L}$  to be a covariant operator mapping  $\lambda$  tensors into  $\lambda + 1$  tensors.

Therefore the following substitutions should be made: the ordinary derivative should be replaced, as before, by the covariant derivative  $\mathcal{D}$ ; the constant regular element  $\lambda K$  should now be regarded as a covariantly constant 1-form, therefore  $\lambda K \mapsto \omega \lambda K$ , with  $\omega$  introduced in (22). As for the term containing the currents  $J_i$ 's, it should be given by 1-forms taking values in the Lie algebra  $\mathcal{G}$ ; their Poisson brackets should be consistent with the requirement that the  $J_i$ 's live in the Riemann surface  $\Sigma$ , and as a consequence it must be provided by the higher genus Kac-Moody algebra (which plays for Kac-Moody the same role as the Krichever-Novikov algebra for Virasoro); a presentation of this algebra is given in [13].

To be definite, let us treat here explicitly the derivation of the NLS hierarchy, the generic

case will follow immediately with trivial modifications.

We have now  $\mathcal{G} \equiv \{H, E_{\pm}\}$ , and commutation relations

$$\begin{aligned} [H, E_{\pm}] &= \pm E_{\pm} \\ [E_+, E_-] &= 2H \end{aligned} \quad (44)$$

$\mathcal{L}$  is given by the covariant operator

$$\mathcal{L} = \mathcal{D} + \lambda H \omega + E_{\pm} J^{\pm} + H J^0 \quad (45)$$

The fields  $J^{\pm}(Q), J^0(Q)$  are 1-forms which can be expanded in their KN-modes as  $J^i = J^i_I \omega^I$ . In a convenient normalization their Poisson brackets algebra can be expressed as

$$\begin{aligned} \{J^0(Q), J^0(Q')\} &= d_{Q'} \Delta(Q', Q) \\ \{J^0(Q), J^{\pm}(Q')\} &= \pm 2 \Delta(Q', Q) J^{\pm}(Q') \\ \{J^+(Q), J^-(Q')\} &= \Delta(Q', Q) J^0(Q') + \frac{1}{2} d_{Q'} \Delta(Q', Q) \end{aligned} \quad (46)$$

and, in terms of the modes:

$$\begin{aligned} \{J^0_I, J^0_J\} &= -\gamma_{IJ} \\ \{J^0, J^{\pm}_J\} &= \pm 2 \alpha^K_{IJ} J^{\pm}_K \\ \{J^+_I, J^-_J\} &= \alpha^K_{IJ} J^0_K - \frac{1}{2} \gamma_{IJ} \end{aligned} \quad (47)$$

where the constants in the r.h.s. are defined in section 2. The diagonalization procedure outlined above in the flat case can be repeated to get the covariant diagonal operator  $\mathcal{L}_{\alpha}$ . The hamiltonians are the covariant versions of those of (41), the integration contour being  $C_{\tau}$ . They are in involution with respect to the higher genus Poisson brackets. At the lowest order we have as hamiltonian densities

$$\begin{aligned} R_0 &= J^0 \\ R_1 &= 2J^+ J^- \\ R_2 &= J^+ \hat{\Delta} J^- - J^- \hat{\Delta} J^+ \end{aligned} \quad (48)$$

In (48,c) the derivative  $\hat{\Delta}$ , covariant with respect to both reparametrizations and the Heisenberg subalgebra (47,a) appears. In general, if  $V^{\lambda}_q$  is a  $q$ -charged  $\lambda$ -tensor, which means that

$$\{J^0(Q), V^{\lambda}_q(Q')\} = q \Delta(Q, Q') V^{\lambda}_q(Q) \quad (49)$$

the covariant derivative  $\hat{\Delta}$  acts as follows

$$\hat{\Delta}V^\lambda_q = (\mathcal{D} - qJ^0)V^\lambda_q \equiv V^{\lambda+1}_q \quad (50)$$

and maps  $V^\lambda_q$  into a  $\lambda + 1$  tensor of definite charge  $q$ .  
Here

$$\hat{\Delta}J^\pm = (\mathcal{D} \mp 2J^0)J^\pm \quad (51)$$

As before, all the hamiltonian densities apart from  $R_0$  belong to the coset with respect to the higher genus Heisenberg algebra (47,a).

The two-components NLS equation is derived from the third hamiltonian  $\oint R_2$  and reads as:

$$\omega^2 \frac{\partial}{\partial t} J^\pm = \pm \hat{\Delta}^2 J_\pm \pm 2(J^+ J^-)J^\pm \quad (52)$$

(the equation relative to  $J^0$  is immediately solved to give  $J^0 \propto \omega$ ).

As in the genus zero case we can derive the scalar form of the NLS hierarchy, which coincides with a particular (non-canonical) reduction of KP. In its more compact form it can be represented as

$$L = \hat{\Delta} + J^- \hat{\Delta}^{-1} J^+ \quad (53)$$

where the pseudodifferential calculus is defined for the totally covariant operator  $\hat{\Delta}$ . The justification of this procedure is contained in [9]. Obviously we can reexpress (53) as a pseudodifferential operator in terms of the derivative  $\mathcal{D}$ ; in this case we maintain the manifest covariance w.r.t. the diffeomorphisms, but not w.r.t. the Heisenberg subalgebra: as a result the computations are more involved and less transparent.

The appearance of a  $\mathcal{W}$  algebra is due to the fact that every hamiltonian density is a polynomial function of the composite fields

$$W_n = \hat{\Delta}^n J^+ \cdot J^- \quad (54)$$

( $n$  non-negative integers) and their covariant ( $\mathcal{D}$ )-derivatives. The same considerations as in [8] hold: the higher order fields ( $n \geq 2$ ) are algebraic functions of  $W_{0,1}$  and their  $\mathcal{D}$ -derivatives: the algebra generated by  $W_0, W_1$  is a higher genus  $\mathcal{W}$  algebra which closes in a rational way, i.e. in the r.h.s. terms proportional to the higher order fields appear. It is convenient to express such algebra in terms of the fields

$$T = \frac{1}{2} J^+ J^- \text{ and } \Psi = \hat{\Delta} J^+ \cdot J^- - \hat{\Delta} J^- \cdot J^+.$$

$T$  plays the role of a stress-energy tensor having (only classically) a vanishing central charge, while  $\Psi$  is a primary field w.r.t.  $T$  having conformal dimension 3. The algebra is explicitly given by

$$\begin{aligned} \{T(Q), T(Q')\} &= 2\Delta^1 \omega(1)T - \Delta \omega(2)T^{(1)} \\ \{T(Q), \Psi(Q')\} &= 3\Delta^1 \omega(1)\Psi - \Delta \omega(2)\Psi^{(1)} \\ \{\Psi(Q), \Psi(Q')\} &= 2\Delta^3 T - 3\Delta^2 \omega(1)T^{(1)} + 2\Delta^1 \omega(2)A - \Delta \omega(3)(A^{(1)} - \frac{1}{2}T^{(2)}) \end{aligned} \quad (55)$$

where the compact notation

$$\begin{aligned}
\mathcal{D}_Q^n \Delta(Q, Q') &\equiv \Delta^n \\
\mathcal{D}^n(T, \Psi) &\equiv T^{(n)}, \Psi^{(n)} \\
\frac{\omega^n(Q)}{\omega^n(Q')} &\equiv \omega(n) \\
A &= 8W_2 - 4\Psi^{(1)} + 4T \cdot T - 2T^{(2)}
\end{aligned} \tag{56}$$

has been used. In the r.h.s. all the fields are evaluated in  $Q'$ . One should notice on the r.h.s. the appearance of the fields  $W_2$ , which is an algebraic functions of  $T, \Psi$ . Since  $T$  and  $\Psi$  are chargeless, it does not matter if the r.h.s. is expressed in terms of the  $\hat{\Delta}$  or the  $\mathcal{D}$  covariant derivatives. It should be pointed out that the existence of the covariantly constant 1-form  $\omega$  allows to have the right tensorial properties for the above algebra in both the points  $Q, Q'$ .

To end this section, let us introduce the free-fields classical Wakimoto representation in higher genus: it provides the NLS extension of the Miura map and allows defining the modified NLS hierarchy (see [9]). It is realized in terms of the 1-form  $p$  and by the coupled (respectively  $(1 \setminus 0)$  tensors)  $\beta \setminus \gamma$ . A convenient way of representing such fields is the following:

$$\begin{aligned}
p(Q) &= \alpha_I \omega^I \\
\beta(Q) &= \beta_I \omega^I \\
\gamma(Q) &= \gamma(Q_0) + \int_{Q_0}^Q \gamma_I \omega^I
\end{aligned} \tag{57}$$

The assumed Poisson brackets in higher genus are the free ones, namely

$$\begin{aligned}
\{p(Q), p(Q')\} &= d_{Q'} \Delta(Q', Q) \\
\{\beta(Q), \gamma(Q')\} &= \Delta(Q', Q)
\end{aligned} \tag{58}$$

and vanishing otherwise.

In terms of the modes they read

$$\begin{aligned}
\{\alpha_I, \alpha_J\} &= -\gamma_{IJ} \\
\{\beta_I, \gamma(Q_0)\} &= A_I(Q_0) \\
\{\beta_I, \gamma_J\} &= -\gamma_{IJ}
\end{aligned} \tag{59}$$

Since  $\gamma$  must be univalued on  $\Sigma$ , the requirements

$$\begin{aligned}
0 = \hat{\gamma}_i &= \gamma_I N^I_i \\
0 = \check{\gamma}_i &= \gamma_I M^I_i
\end{aligned} \tag{60}$$

should be imposed ( $N^I_i, M^I_i$  are given in (12,13)).

It is a remarkable feature of the Heisenberg algebra on higher genus that  $\hat{\gamma}_i, \check{\gamma}_i$  have vanishing Poisson brackets with respect to any element of the algebra, and therefore (60) can be imposed without further constraints. The algebra (46) can be reproduced through the

identification

$$\begin{aligned}
J^+(Q) &= \beta(Q) \\
J^0(Q) &= (p - 2\beta\gamma)(Q) \\
J^-(Q) &= (p\gamma - \beta\gamma^2 + \tfrac{1}{2}d\gamma)(Q)
\end{aligned} \tag{61}$$

as it can be straightforwardly checked. The modified NLS equation can be derived, by inserting (61) in  $R_2$ : we obtain the coupled system for  $\beta, \gamma$

$$\begin{aligned}
\omega^2 \dot{\beta} &= \mathcal{D}^2 \beta + 2\beta^2 \mathcal{D} \gamma - 2\beta^3 \gamma^2 \\
\omega^2 \dot{\gamma} &= \mathcal{D}^2 \gamma - 2\gamma^2 \mathcal{D} \beta - 2\gamma^3 \beta^2
\end{aligned} \tag{62}$$

which generalizes the genus zero results. Here the constraint  $J^0 = 0$ , which is consistent with the equations of motion, is set to get rid of the field  $p$  in the above equations.

## Conclusions

The KP hierarchy and its reductions have recently been investigated in connection with the matrix model formulation of the 2-dimensional gravity. Starting from the standard KP hierarchy there are several possible ways of generalizing it: by enlarging its algebraic setting allowing supersymmetry; by quantizing it (it could be speculated that such quantum versions are linked to some string-field theory); by  $q$ -deforming it, which leads to the connection with quantum group. In this paper I have shown an appropriate framework to put it on Riemann surfaces, which became popular in physicists' literature due to the Polyakov formulation of the string perturbation theory. It is likely that the formulation here proposed be relevant for investigating the genus expansion of matrix models. This will be left for future investigations.

## Acknowledgements

I wish to express my thanks to Prof. M. Tonin and to the Physics Department of the Padua University for their kind hospitality.

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